SURFACES WITH RATIONAL SINGULARITIES

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These are notes for a talk I gave in a graduate student seminar on semi-orthogonal decompositions in derived categories. Any mistakes are my own.

Recall that a normal surface with rational singularities is a surface X with a proper birational map $\pi : Y \to X$ from a regular scheme Y such that $\pi_*\mathcal{O}_Y \cong \mathcal{O}_X$ (in the derived sense). For this talk, we will restrict attention to rational surfaces, but the arguments work in more generality (see [KKS20]). We will also only look at such surfaces over an algebraically closed field of characteristic 0.

We would like to say something intelligent about semi-orthogonal decompositions of $D^b(X)$. But we don't want just any semi-orthogonal decomposition. Rather, we would like the constituents to be something nice. Asking the constituents to be point-like objects is perhaps too ambitious, so we ask instead that they behave like the bounded derived category of local finite-dimensional (perhaps non-commutative) algebras over a field.

We will describe a process for doing this systematically in good cases. The key idea will be to take a semi-orthogonal decomposition of the resolution Y, and descend it down to X by effectively collapsing $\mathcal{O}_E(-1)$ for E exceptional divisors in Y. This will only work when $\mathcal{O}_E(-1)$ lies in $\tilde{\mathcal{A}}_i$ for some constituent of the semi-orthogonal decomposition of Y.

Preliminaries

We begin with more on normal rational surface with rational singularities. Let Y be a resolution of X, i.e. $\pi : Y \to X$ is some proper map which arises as blowups along points and $\pi_*\mathcal{O}_Y \cong \mathcal{O}_X$. In this case, the exceptional locus is a disjoint union of trees of smooth rational curves with negative definite intersection matrix.

We have two maps: $\pi^* : D^-(X) \to D^-(Y)$ and $\pi_* : D^-(Y) \to D^-(X)$, and by the projection formula, we have

$$\pi_* \circ \pi^* \cong \mathrm{id}_{D^-(X)}$$

So, we get a semi-orthogonal decomposition,

$$D^-(Y) = \langle \mathrm{Ker}(\pi_*), \pi^* D^-(X) \rangle$$

(essentially, pullback has a left-adjoint in our case).

Definition: Suppose we have a semi-orthogonal decomposition of $D^b(Y) = \langle \tilde{\mathcal{A}}_1, ..., \tilde{\mathcal{A}}_n \rangle$ such that for each irreducible component E of the exceptional locus of π, $\mathcal{O}_E(-1) \in \tilde{\mathcal{A}}_i$ for some i. We then say such a semi-orthogonal decomposition is *compatible* with π.

Lemma: If we have two irreducible components in E, E' of the exceptional locus with $\mathcal{O}_E(-1) \in \tilde{\mathcal{A}}_i$ and $\mathcal{O}_{E'}(-1) \in \tilde{\mathcal{A}}_j$ for $i \neq j$, then $E \cap E' = \emptyset$, and so $\mathcal{O}_E(-1)$ and $\mathcal{O}_{E'}(-1)$ are completely orthogonal. This has to do with Riemann-Roch, i.e. the alternating sum of Ext of these two sheaves being equal to the intersection number of E and E'.

Now, the idea is that from the two-piece semi-orthogonal decomposition of $D^-(Y) = \langle \mathrm{Ker}(\pi_*), \pi^*(D^-(X)) \rangle$, we notice that $\mathcal{O}_E(-1)$ is in $\mathrm{Ker}(\pi_*)$ (and, in fact, generates it).

So, we would like to kill the kernel piece in the derived category and recover the bounded above derived category for X.

So, let us assume that X is additionally projective. Then, let us take any old semi-orthogonal decomposition of $D^b(Y) = \langle \tilde{\mathcal{A}}_1, ..., \tilde{\mathcal{A}}_n \rangle$ compatible with π. Then, by definition, $\mathcal{O}_E(-1)$ lies in one of the $\tilde{\mathcal{A}}_i$. We can then partition the exceptional divisor's irreducible components into pieces D_i so that D_i contains those irreducible components E of the exceptional locus such that $\mathcal{O}_E(-1) \in \tilde{\mathcal{A}}_i$.

The constituents of this semi-orthogonal decomposition will be admissible, so we will be able to extend them to a semi-orthogonal decomposition of the bounded above derived category,

$$D^-(Y) = \langle \tilde{A}_1^-, ..., \tilde{A}_n^- \rangle$$

so that $\tilde{\mathcal{A}_i^-} \cap D^b(Y) = \tilde{\mathcal{A}}_i$.

We arrive at the following important lemma which explains how each $\tilde{\mathcal{A}}_i \cap \mathrm{Ker}(\pi_*)$ is built.

Lemma: $\tilde{\mathcal{A}_i^-} \cap \mathrm{Ker}(\pi_*)$ is "additively" generated by $\mathcal{O}_E(-1)$ for those $E \in D_i$. Moreover, for any $\mathcal{F} \in \mathrm{Ker}(\pi_*)$, there is a canonical decomposition $\mathcal{F} = \bigoplus_{i=1}^n \mathcal{F}_i$ where $\mathcal{F}_i \in \tilde{\mathcal{A}_i^-} \cap \mathrm{Ker}(\pi_*)$. Furthermore, $\mathrm{Ext}^\bullet(\tilde{\mathcal{A}_i^-}, \tilde{\mathcal{A}_j^-} \cap \mathrm{Ker}(\pi_*)) = 0$ if either $i < j$ and π is crepant along D_j, or if $i > j$.

Frustratingly, the speaker has left out a proof of this lemma.

Now, let $\tilde{\alpha}_i : D^-(Y) \to D^-(Y)$ be the projection functor onto the $\tilde{\mathcal{A}_i^-}$ piece (so that the essential image of the functor is $\tilde{\mathcal{A}_i^-}$). We then have the following theorem:

Theorem:

(i) $\langle \mathcal{A}_1^-, ..., \mathcal{A}_n^- \rangle$ with projection functors given by $\alpha = \pi_* \circ \tilde{\alpha}_i \circ \pi^*$ forms a semi-orthogonal decomposition of $D^-(X)$.

(ii) $\pi^*(\mathcal{A}_i) \subset \langle \tilde{\mathcal{A}_i^-}, \tilde{\mathcal{A}_{i+1}^-} \cap \mathrm{Ker}(\pi_*), ..., \tilde{\mathcal{A}_n^-} \cap \mathrm{Ker}(\pi_*) \rangle$

(iii) If, in addition, π were crepant along D_j for all $j > i$, then $\pi^*(\mathcal{A}_i) \subset \tilde{\mathcal{A}}_i$.

Proof Sketch: We first sketch proofs of (ii) and (iii). The idea is to take some element $\mathcal{F} \in \tilde{\mathcal{A}_i^-}$ and look at the triangle induced by the counit map

$$\pi^*\pi_*\mathcal{F} \to \mathcal{F} \to \mathcal{F}'$$

Then, we note that $\mathcal{F}' \in \mathrm{Ker}(\pi_*)$, so by the above lemma, we can decompose it into a direct sum $\mathcal{F}' = \bigoplus_{i=1}^n \mathcal{F}'_i$ with $\mathcal{F}'_i \in \tilde{\mathcal{A}}_i \cap \mathrm{Ker}(\pi_*)$. Now, you check that $\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{F}'_j) = 0 = \mathrm{Ext}^\bullet(\pi^*\pi_*\mathcal{F}, \mathcal{F}'_j)$ for $j < i$, and so $\mathrm{Hom}(\mathcal{F}', \mathcal{F}'_j)$ for $j < i$. Thus, $\mathcal{F}' \in \langle \tilde{\mathcal{A}_i^-}, \tilde{\mathcal{A}_{i+1}^-} \cap \mathrm{Ker}(\pi_*), ..., \tilde{\mathcal{A}_n^-} \cap \mathrm{Ker}(\pi_*) \rangle$, and so is $\mathcal{F}$, which means so is $\pi^*\pi_*(\mathcal{F})$, and since $\mathcal{A}_i^-$ is mapped onto surjectively by $\tilde{\mathcal{A}_i^-}$, we win.

A similar argument shows, using the above lemma, that if we assume π is crepant along D_j for $j > i$, then you get the stronger statement of part (iii).

We now turn to a proof of the first part. The fact that $\mathcal{A}_i^-$ is triangulated is purely formal. You take some $\mathcal{G} \in \mathcal{A}_i^-$, and then look at the triangle,

$$\mathcal{G}' \to \pi^*\mathcal{G} \to \tilde{\alpha}_i(\pi^*\mathcal{G})$$

and get that $\pi_*\mathcal{G}' = 0$ and use that to conclude that α_i is the identity on $\mathcal{A}_i^-$. Then, it is a well-known fact in the industry that $\mathcal{A}_i^-$ is triangulated.

Semi-orthogonality of the pieces is even easier using adjunction. Now, suppose we have $\mathcal{F} \in D^-(X)$. Then, looking at $\pi^*\mathcal{F} = \mathcal{F}_Y$, we have a chain for $\mathcal{F}_Y$ whose cones are $\tilde{\alpha}_i(\mathcal{F}_Y)$.

Pushing forward the chain, we get a chain for $\mathcal{F}$ whose cones are $\pi_*(\tilde{\alpha}_i(\mathcal{F}_Y)) = \alpha_i(\mathcal{F})$, as needed.

REFERENCES (INFORMAL)

The references are the same as those listed on the website for the seminar, with the relevant sections used.